Chapter 6: Oscillations

6.1 Introduction

Oscillations are periodic fluctuations around a stable equilibrium point. This unit is critical for engineers because almost every physical system, whether a skyscraper, a vehicle suspension, or an AC circuit, has a natural frequency at which it wants to vibrate. Understanding how to model, damp, or drive these vibrations is fundamental to structural stability and signal processing.

6.2 Types of Oscillations

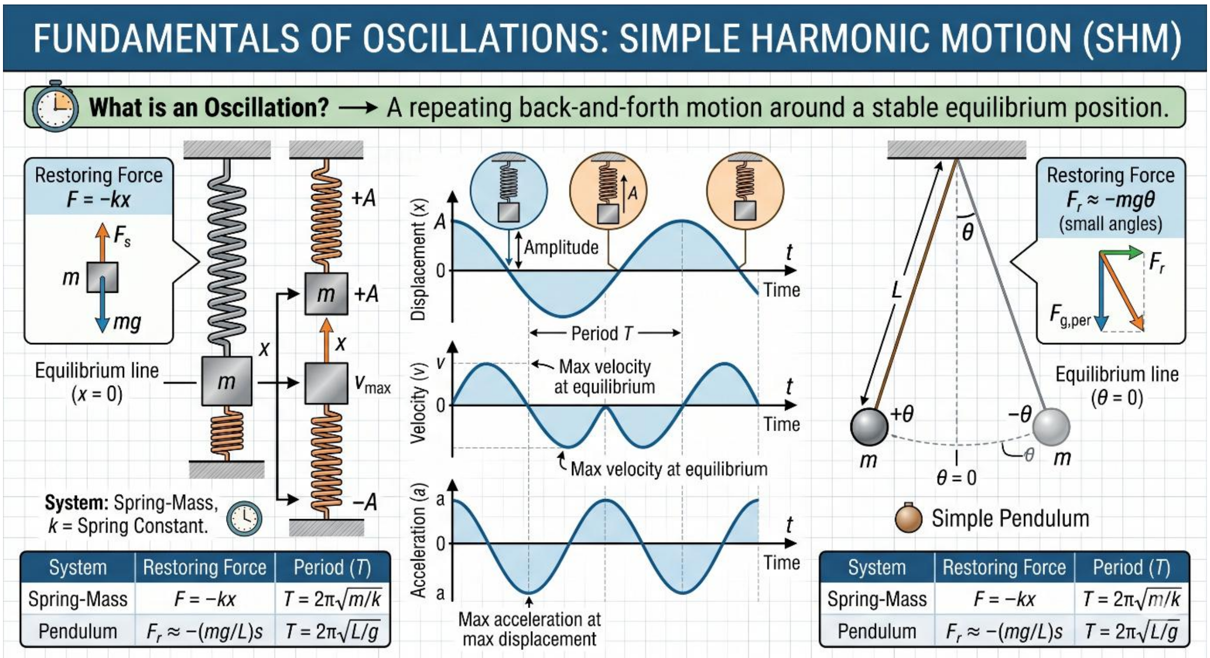

Figure 6.1: Illustration for Oscillations and SHM

Simple Harmonic Motion (SHM)

SHM is the simplest model of oscillation where the system contains no friction (conservative system).

The Condition for SHM: Restoring force is linear and opposite to displacement (Hooke's Law): $F_{restoring} = -kx$

Equation of Motion: Applying Newton's 2nd Law ($F = ma$):

$$m\frac{d^2x}{dt^2} = -kx \implies \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Defining the **natural angular frequency** $\omega_n = \sqrt{\frac{k}{m}}$:

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

The Solution: The displacement as a function of time is sinusoidal:

$$x(t) = Acos(\omega_n t + \phi)$$

A: Amplitude (determined by initial energy)

ϕ: Phase constant (determined by initial position/velocity)

Energy in SHM

In an ideal SHM system, mechanical energy is conserved and continuously converts between potential and kinetic forms.

- **Potential Energy (U):** Stored in the spring.

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2cos^2(\omega_n t + \phi)$$

Kinetic Energy (K): Possessed by the moving mass.

$$K(t) = \frac{1}{2}kv^2 = \frac{1}{2}m\omega_n^2 A^2 sin^2(\omega_n t + \phi)$$

Total Energy (E):

$$E = K + U = \frac{1}{2}kA^2 = constant$$

Damped Harmonic Motion

Real systems lose energy to their environment (friction, air resistance, viscosity). This is modelled by a damping force proportional to velocity.

The Damping Force: $F_d = -bv = -b\frac{dx}{dt}$

where b is the damping constant.

Equation of Motion: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

Regimes of Damping: The behavior is determined by comparing the damping term to the restoring force term. We often define the damping parameter $\gamma = \frac{b}{2m}$.

Underdamped ($\omega_n > \gamma$): The system oscillates with decaying amplitude. The envelope of decay is exponential:$Ae^{-\gamma t}$.

Critically Damped ($\omega_n = \gamma$): The system returns to equilibrium as fast as possible *without* overshooting or oscillating. **Crucial for engineering applications like vehicle shock absorbers.**

Overdamped ($\omega_n < \gamma$): The damping is so strong that the system returns to equilibrium very slowly (sluggishly).

Forced Oscillations and Resonance

Occurs when an external periodic driver acts on the system: $F_{ext}(t) = F_0 cos\,(\omega_{dr} t)$.

Equation of Motion: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 cos\,(\omega_{dr} t)$

Steady State Solution:

The system eventually oscillates at the driving frequency (ω_{dr}), not its natural frequency.

Resonance:

When the driving frequency matches the natural frequency ($\omega_{dr} \approx \omega_n$), the amplitude of oscillation peaks dramatically.

- **Q-Factor:** A measure of the "sharpness" of resonance. High Q means low damping and sharp resonance (e.g., a tuning fork). Low Q means high damping and broad resonance.

Applications of Oscillations in Engineering

Mechanical Systems

- **Suspension Systems**: Car shock absorbers are designed to be critically damped so that after a bump, the car returns to a steady level immediately without bouncing.
- **Structural Engineering:** Skyscrapers incorporate "Tuned Mass Dampers" massive pendulums near the top floor to counter oscillations caused by wind or earthquakes.

Electrical Systems (The RLC Circuit)

There is a direct mathematical equivalence between mechanical and electrical oscillations.

Inductor (L): Acts like Mass (m) – resists changes in current (velocity).

Capacitor (C): Acts like a Spring (k) – stores potential energy in an electric field.

Resistor (R): Acts like the Damping constant (b) – dissipates energy as heat.

Application: Radio tuners work by adjusting capacitance to create **Resonance** at a specific broadcast frequency, filtering out all other signals.

Optical and Atomic Systems

- **Quartz Crystals:** Used in watches and computers, these vibrate at a precise resonant frequency due to the piezoelectric effect, acting as the system's "heartbeat" or clock.
- **Molecular Vibrations:** Chemical bonds act like springs. Infrared spectroscopy uses resonance to identify molecules based on their vibrational frequencies.

6.6 Worked Examples

Example 1 – Mass-Spring Oscillator

A 0.5 kg mass is attached to a spring of stiffness 200 N/m. Find the frequency of oscillation.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{0.5}} = \sqrt{400} = 20\ rads^{-1}$$

$$f = \frac{\omega}{2\pi} = \frac{20}{6.283} \approx 3.18Hz$$

So, the system oscillates about 3 times per second.

Example 2 – Pendulum

A simple pendulum of length L=1.0 m. Find the period of small oscillations.

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1}{9.81}} \approx 2.01s$$

So, it swings back and forth every ~2 seconds.

Real-Life Engineering Applications

- **Mechanical Systems:**
  - Car suspensions (damping).

    - Clock pendulums (timekeeping).
    - Seismographs (earthquake detection).

- **Electrical Systems (AC Circuits):**
    - LC circuits oscillate between electric and magnetic energy.
    - Resonance used in tuning radios and wireless communication.
- **Civil Engineering:**
    - Buildings/bridges designed to avoid resonance with wind or earthquake frequencies.
- **Medical Applications:**
    - Ultrasound imaging uses oscillations of high-frequency sound waves.
    - MIR scanner gives a detailed three-dimensional view of the body.
    - Heart pacemakers rely on periodic electrical oscillations.

Activity: Practical CUMSM 110 No.2

Simple Harmonic Oscillations on a Pendulum

Objective: To verify that the period of a simple pendulum is independent of mass and amplitude (for small angles) and depends only on length (L) and gravity (g).

Key Relationship to Verify: $T = 2\pi\sqrt{\frac{L}{g}}$

Plotting: Plot T^2 vs L. The slope of this graph can be used to experimentally determine the acceleration due to gravity (g).

Exercises

1. At time *t* = 0, the displacement *x*(0) of the block in a linear oscillator like that of Fig. 2 is $-8.50\ cm$. The block's velocity *v*(0) then is $-0.93\ m/s$, and its acceleration *a*(0) is $45.0\ m/s^2$. Determine,

(i) the angular frequency of this system,

(ii) the phase constant and

(iii) the amplitude of this system.

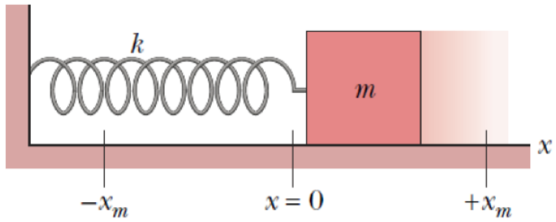

Fig 2.

2. A block weighing 20 N oscillates at one end of a vertical spring for which $k = 100\ N/m$; the other end of the spring is attached to a ceiling. At a certain instant the spring is stretched $0.30\ m$ beyond its relaxed length (the length when no object is attached) and the block has zero velocity.

    a) What is the net force on the block at this instant?
    b) What is the amplitude and,
    c) period of the resulting simple harmonic motion?
    d) What is the maximum kinetic energy of the block as it oscillates?

3. Consider the simplified single-piston engine in Figure 3. Assuming the wheel rotates with constant angular speed, explain why the piston rod oscillates in simple harmonic motion. [4]

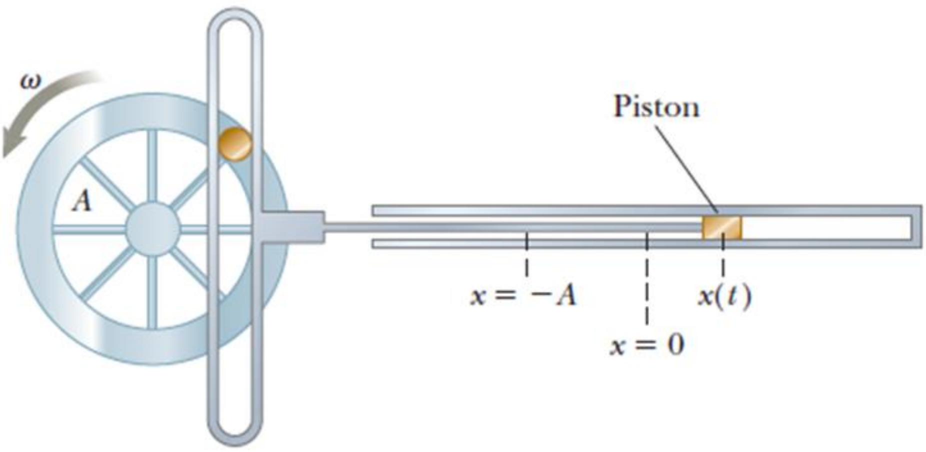

Figure 3